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Full stability-exponent placement in chaotic systems

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We present an algorithm to allow full placement of all stability exponents (Poincaré or Lyapunov) in a controlled chaotic system. The linear quadratic regulator of classical control theory is recast to allow specification of the controlled system Lyapunov exponents and initial and final principal dynamical directions. In the process, a positive definite functional of the control is minimized. The boundary value problem that must be solved is linear, and converges in one iteration. Successful results are reported, applying the method to the Duffing oscillator, the Lorenz system, and the restricted problem of three bodies, for both periodic orbits and general trajectories.

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I. INTRODUCTION

Control theory for constant coefficient systems is very well developed, but the situation is less clear in the case of time dependent systems. Time dependent linear systems arise when a general trajectory is linearized to study its stability. Periodic orbits give rise to periodic coefficient linear systems.

Most current work on controlling chaotic systems follows the seminal work of Ott, Grebogi, and Yorke [1], based on applying impulsive control at the crossing of a surface of section. A review of this area was given recently by Shinbrot, Grebogi, Ott, and Yorke [2]. Attempts to dictate the Lyapunov exponents of a controlled system include Wiesel [3]. In that paper the modal decomposition of the time dependent linear system was used to dictate one Lyapunov exponent at a time. This method was extended and applied to the reentry of a spacecraft from orbit [4]. In this paper we report a feasible method for setting *all* of the controlled system's Lyapunov exponents, and as well setting the principal dynamical directions.

II. LINEAR QUADRATIC REGULATORS

The equations of motion for a nonlinear system can be written

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, \mathbf{U}, t), \tag{1}$$

where the state variables are written as the vector $\mathbf{X}$, and certain of the system parameters accessible to our manipulation are termed the control vector $\mathbf{U}$. When this is linearized about a known solution $\mathbf{X}_0(t)$ and a nominal control history $\mathbf{U}_0(t)$, we have the variational problem

$$\begin{aligned}\dot{\mathbf{x}} &= \left.\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right|_{\mathbf{X}_0,\mathbf{U}_0} \mathbf{x} + \left.\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right|_{\mathbf{X}_0,\mathbf{U}_0} \mathbf{u} \\ &= A(t)\mathbf{x} + B(t)\mathbf{u}. \end{aligned}\tag{2}$$

This is termed the "closed loop system," that is, the system with the control term operating. Here $\mathbf{x} = \mathbf{X}(t) - \mathbf{X}_0(t)$ and $\mathbf{u} = \mathbf{U}(t) - \mathbf{U}_0(t)$ are the first order deviations of the state and control from their nominal programs. It is our goal to choose $\mathbf{u}(t)$ to achieve specified stability properties for the solution $\mathbf{x}_0(t)$, thus ensuring that $\mathbf{X}_0(t)$ is linearly stable.

One technique for controlling such systems is the linear quadratic regulator, or LQR controller; see, e.g., Bryson and Ho [5]. Although most commonly used on constant coefficient systems, it is applicable to time dependent systems as well. This technique chooses a control to minimize the cost function

$$\mathcal{J} = \frac{1}{2}\left(\mathbf{x}^T S_f \mathbf{x}\right)\Big|_{t=t_f} + \frac{1}{2}\int_{t_0}^{t_f} \left(\mathbf{u}^T C \mathbf{u} + \mathbf{x}^T D \mathbf{x}\right) dt \tag{3}$$

over the time interval $t_0 \le t \le t_f$. The matrices C, D, and S_f are symmetric and positive definite, and penalize, respectively, the use of control, deviations from the nominal trajectory, and deviations from the nominal trajectory final conditions. They are usually chosen as diagonal matrices. The latter two weighting matrices are needed, since until recently stability theory for general time-dependent linear systems was not well known. The choice of the weighting matrices is a problem with this method. While arguments about maximum "acceptable" values of $\mathbf{u}$, $\mathbf{x}(t)$, and $\mathbf{x}(t_f)$ can be made, a well known fact about LQR controllers is that *any* linear control solution, stable or not, is the solution of an LQR problem with some weighting matrices. Also, solutions minimizing (3) for "reasonable" weighting matrices exist which are actually *unstable.*

In this work we will dispense with the weighting matrix D for the state, and will replace merely discouraging state deviations from the nominal trajectory with explicit stability criteria for the "closed loop" system. The cost function then becomes

$$\mathcal{J} = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{u}^T C \mathbf{u} dt, \tag{4}$$

subject to (2) as differential constraints. We proceed by appending the constraints to the cost function with a vector of Lagrange multipliers $\boldsymbol{\gamma}$, producing the control Hamiltonian as

$$\mathcal{H} = \frac{1}{2}\mathbf{u}^T C\mathbf{u} + \boldsymbol{\gamma}^T (A\mathbf{x} + B\mathbf{u}). \tag{5}$$

Hamilton's equations for the optimal control problem are

$$\dot{\mathbf{x}} = \frac{\partial\mathcal{H}}{\partial\boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = -\frac{\partial\mathcal{H}}{\partial\mathbf{x}}. \tag{6}$$

The first of these reproduce (2), while the second becomes

$$\dot{\boldsymbol{\gamma}} = -A^T\boldsymbol{\gamma}. \tag{7}$$

In addition, the *optimality condition*

$$0 = \frac{\partial\mathcal{H}}{\partial\mathbf{u}} = C\mathbf{u} + B^T\boldsymbol{\gamma} \tag{8}$$

specifies that the solution minimizes (4). It can be immediately solved to yield the *control law*

$$\mathbf{u} = -C^{-1}B^T\boldsymbol{\gamma}. \tag{9}$$

Equations (2), (7), and (9) must be solved as a boundary value problem. Typically, we will have initial conditions on $\mathbf{x}(t_0)$, and are free to impose one other set of boundary values. We make the choice $\boldsymbol{\gamma}(t_f) = S_f\mathbf{x}(t_f)$, which seems the same as in (3) where this statement is a direct consequence of the formulation of the problem. But the matrix S_f, usually specified *ad hoc* as a symmetric, positive definite matrix, will not be so here. We will find that S_f is the key to imposing stability criteria, and will leave its choice for the next section.

In classical LQR theory it is common to assert that the variables $\mathbf{x}$ and $\boldsymbol{\gamma}$ are related by the matrix equation

$$\boldsymbol{\gamma}(t) = S(t)\mathbf{x}(t). \tag{10}$$

Differentiating the above with respect to time, and repeatedly using (2), (7), (9), and (10) leads to

$$\dot{S}\mathbf{x} + SA\mathbf{x} - SBC^{-1}B^T S\mathbf{x} = -A^T S\mathbf{x}. \tag{11}$$

Since we wish this transformation to be true for all trajectory deviations $\mathbf{x}(t)$, we are led to the matrix Riccati equation

$$\dot{S} = -SA - A^T S + SBC^{-1}B^T S. \tag{12}$$

Notice that this equation can be propagated independent of either the closed loop linear system (2) or the Lagrange multiplier equations (7). Also, with S_f specified *ad hoc*, final conditions are already known. This leads to the "dual sweep" algorithm: Eq. (12) is integrated backwards from $S = S_f$ to the initial time, and then (10) yields $\boldsymbol{\gamma}(t_0)$, enabling (2) and (7) to be integrated forward. Alternately, the closed loop system can be rewritten as

$$\dot{\mathbf{x}} = (A + BG)\,\mathbf{x}, \tag{13}$$

where the *gain matrix* $G(t)$ is given by

$$G(t) = -C^{-1}B^T S. \tag{14}$$

The gain matrix as a function of time is all that the real-time control system needs to actually "fly" the trajectory.

III. STABILITY CONDITIONS

We wish to replace the *ad hoc* choice of S_f with explicit stability conditions on the closed loop dynamics. [The weighting matrix $C(t)$ must still be chosen by the control designer, and it should be symmetric and positive definite. It specifies the degree to which individual control terms are to be discouraged, and that is a human decision.] Lyapunov exponents specify the stability of a trajectory arc. The "open loop" Lyapunov exponents (that is, without control) are easy to determine. The open loop system is $\dot{\mathbf{x}} = A\mathbf{x}$, and this can be solved numerically by integrating the state transition matrix $\Phi_{x,O}$, which obeys

$$\dot{\Phi}_{x,O} = A\Phi_{x,O}, \quad \Phi_{x,O}(t_0) = I, \tag{15}$$

where I is the identity matrix. This can then be factored at $t = t_f$ into its singular vectors and values as

$$\Phi_{x,O}(t_f) = \mathcal{U}_O\mathcal{W}_O\mathcal{V}_O^T. \tag{16}$$

The open loop Lyapunov exponents are then given by

$$\lambda_{i,O} = \frac{1}{t_f - t_0}\ln w_{i,O}, \tag{17}$$

where the $w_{i,O}$ are the elements of the diagonal matrix $\mathcal{W}_O$. The subscript "O" specifies the open loop system. It would be more desirable to calculate (and specify) the Lyapunov exponents of the closed loop system.

The state transition matrix for the closed loop system could be similarly factored as

$$\Phi_{x,C}(t_f) = \mathcal{U}_C\mathcal{W}_C\mathcal{V}_C^T, \tag{18}$$

where the closed loop Lyapunov exponents $\lambda_{i,C}$ give the elements $w_{i,C} = \exp\left(\lambda_{i,C}(t_f - t_0)\right)$. The orthonormal matrix $\mathcal{V}_C$ gives the directions in space on a unit sphere at $t = t_0$ which evolve into the directions $\mathcal{U}_C$ at $t = t_f$ of an ellipsoid whose axis lengths are $w_{i,C}$. These axis lengths are functions of the closed loop Lyapunov exponents $\lambda_{i,C} = \ln w_{i,C}/(t_f - t_0)$. It is our desire to be able to specify *all of the above in advance.* In the control theory of constant-coefficient systems, this is termed "full exponent–vector placement." Of course, without good reason to do otherwise, we are usually only interested in specifying new closed loop Lyapunov exponents for the system, and will leave the principal dynamical direction vectors unaltered,

$$\mathcal{U}_O = \mathcal{U}_C, \quad \mathcal{V}_O = \mathcal{V}_C. \tag{19}$$

However, the method to follow allows the choice of all three matrices. Then, with $\mathcal{U}_C$, $\mathcal{V}_C$, and the $\lambda_{i,C}$ specified, the desired closed loop matrix $\Phi_{x,C}$ is determined.

Using the control law (9), the closed loop system and Lagrange multiplier equations of motion can be rewritten as

$$\dot{\mathbf{x}} = A\mathbf{x} - BC^{-1}B^T\boldsymbol{\gamma}, \quad (20)$$
$$\dot{\boldsymbol{\gamma}} = -A^T\boldsymbol{\gamma}. \quad (21)$$

Then, calculating partial derivatives with respect to the initial conditions $\mathbf{x}(t_0)$ gives

$$\frac{d}{dt}\frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)} = A\frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)} - BC^{-1}B^T\frac{\partial \boldsymbol{\gamma}(t)}{\partial \mathbf{x}(t_0)}, \quad (22)$$
$$\frac{d}{dt}\frac{\partial \boldsymbol{\gamma}(t)}{\partial \mathbf{x}(t_0)} = -A^T\frac{\partial \boldsymbol{\gamma}(t)}{\partial \mathbf{x}(t_0)}. \quad (23)$$

The first quantity is just the closed loop state transition matrix

$$\Phi_{x,C}(t) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)}, \quad (24)$$

while the second quantity is abbreviated as $\Phi_{\boldsymbol{\gamma}}(t) = \partial\boldsymbol{\gamma}(t)/\partial\mathbf{x}(t_0)$. The differential equations then become

$$\frac{d}{dt}\Phi_{x,C} = A\Phi_{x,C} - BC^{-1}B^T\Phi_{\boldsymbol{\gamma}}, \quad (25)$$
$$\frac{d}{dt}\Phi_{\boldsymbol{\gamma}} = -A^T\Phi_{\boldsymbol{\gamma}}. \quad (26)$$

Since we have initial conditions $\Phi_{x,C}(t_0) = I$ specified by (24), the problem becomes finding initial conditions $\Phi_{\boldsymbol{\gamma}}(t_0)$ to achieve given final conditions $\Phi_{x,C}(t_f)$. Notice that the open loop case is obviously $\Phi_{\boldsymbol{\gamma}}(t_0) = \{0\}$, where $\{0\}$ is the zero matrix.

The above equations are a *linear* boundary value problem. This problem can be solved numerically by integrating Eqs. (25) and (26) once to obtain the open loop solution, and then N^2 more times to obtain the partial derivatives $\partial\Phi_{x,C}(t_f)/\partial\Phi_{\boldsymbol{\gamma}}(t_0)$ numerically. Since this is a linear problem in $\Phi_{\boldsymbol{\gamma}}(t_0)$, the numerical partial derivatives are exact to within roundoff error. One iteration of a Newton-Rhapson scheme

$$\Phi_{\gamma}(t_0) = \left\{\frac{\partial\Phi_{x,C}(t_f)}{\partial\Phi_{\gamma}(t_0)}\right\}^{-1}[\Phi_{x,C}(t_f) - \Phi_{x,O}(t_f)] \quad (27)$$

will converge to the desired values of $\Phi_{x,C}(t_f)$ and $\Phi_{\boldsymbol{\gamma}}(t_0)$. [Note that the quantities above are suitably partitioned, so that Φ_i are stored as N^2 vectors, and $\partial\Phi_{x,C}(t_f)/\partial\Phi_{\gamma}(t_0)$ is an N^2 by N^2 matrix. This matrix must be invertible: this is termed the *controllability condition.*]

Notice, taking partial derivatives of (10) with respect to the initial conditions $\mathbf{x}(t_0)$, that

$$\Phi_{\gamma}(t) = S(t)\Phi_{x,C}(t). \quad (28)$$

Then knowledge of $\Phi_{\boldsymbol{\gamma}}(t_0)$ gives $S(t_0) = \Phi_{\boldsymbol{\gamma}}(t_0)$, since $\Phi_{x,C}(t_0) = I$. Then integrating the Riccati equation (12) forward, the value of $S(t_f) = S_f$ is obtained. So S_f is specified implicitly by the choice of the Lyapunov exponents and principal dynamical directions for the closed loop system. The matrix S_f is not likely to be a positive definite matrix, and is very unlikely to be diagonal. This means that we have not found a true minimum of (3) in $\mathbf{x}, \mathbf{u}$ space. However, if $C(t)$ is positive definite, then we do have a true minimum of (4) in the control directions. The requirement to minimize an *ad hoc* positive-definite function of the state, both within and at the end of the time interval, has been discarded, and replaced with complete freedom to specify the closed loop Lyapunov exponents and principal dynamical directions. Knowing $S(t_0)$, the Riccati equation can be integrated forwards defining $S(t)$ and the gain matrix $G(t)$ through (14). With the gain matrix defined, the closed loop system in the form (13) can be used in real time to control the system.

Much of modern work on control of chaotic systems actually attempts to stabilize unstable periodic orbits imbedded within the attractor. The known pole placement algorithm for periodic systems, Calico and Wiesel [6], has the same problem as that of [3]: only one exponent at a time may be placed. In Floquet theory, the state transition matrix for a periodic orbit is decomposed as

$$\Phi_x(t) = E(t)\exp(\Lambda(t - t_0))E^{-1}(t_0), \quad (29)$$

where $E(t)$ is a matrix periodic with the same period as the underlying orbit, and the entries of the diagonal matrix Λ,λ_i are termed the Poincaré exponents. Both the Poincaré exponents and the matrix $E(t)$ are possibly complex quantities. Evaluating (29) at the end of one period $t_f - t_0 = \tau$ gives the monodromy matrix, and using the fact that $E(t)$ is periodic yields

$$\Phi_x(\tau) = E(t_0)\exp(\Lambda\tau)E^{-1}(t_0). \quad (30)$$

Open loop stability exponents are easily calculated from the above using a standard eigenvalue-eigenvector routine.

But (30) can also be used to *dictate* both the closed loop Poincaré exponents and the closed loop modal matrix $E(t_0)$. [Poincaré exponents are much to be preferred over Lyapunov exponents for periodic orbits. The principal dynamical direction matrices $\mathcal{U}$, $\mathcal{V}$ do *not* close upon themselves at the end of one period, while this is automatic with Floquet modal vectors $E(t)$.] We are free to change one or more offensive Poincaré exponents, probably leaving the modal matrix $E(t_0)$ unchanged, and then (30) will supply the desired closed loop state transition matrix $\Phi_{x,C}(\tau)$. The LQR theory discussed earlier then goes through with no further modifications. Once the Poincaré exponents are set over one period, the trajectory remains linearly stable forever.

IV. NUMERICAL EXPERIMENTS

The algorithm of the preceding section has been applied to several standard problems. To begin, we have considered the Lorenz problem [7] modified with a control input,

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= -xz + ax - y + u, \quad (31)\\ \dot{z} &= xy - bz.\end{aligned}$$

Parameter values were $\sigma = 16$, $a = 40$, and $b = 4$, while $u(t)$ is the scalar control input. The matrix $C = 1$ becomes a scalar, and all integrations began at initial conditions on the attractor $x = 2.426\ 881\ 355\ 528$, $y = 2.577\ 259\ 040\ 064$, and $z = 26.689\ 700\ 667\ 84$. The control appears on the state y following Chen and Chou [8]. Figure 1 shows both the open loop and closed loop Lyapunov exponents as a function of t_f, with the open loop values shown as solid curves. Values of the open loop Lyapunov exponents for this trajectory as $t_f \to \infty$ are 1.37, 0, and −22.37. The open loop Lyapunov exponents are clearly approaching these values before $t_f \approx 1.2$, at which point the smallest exponent can no longer be reliably calculated in double precision arithmetic.

Closed loop Lyapunov exponents are shown in Fig. 1 as dots, with each dot representing one control system solution. Closed loop exponents were chosen to be −1, −2, and −22, with no alteration to the principal dynamical directions. The algorithm works very well until $t_f \approx 0.8$, after which the desired value for the smallest exponent is not achieved. The reason for both of these problems is the same: the elements of the $\mathcal{W}$ matrix now span over 13 orders of magnitude, and the smallest of the $\lambda_{i,C}$ are incorrect. However, the larger two closed loop Lyapunov exponents are correctly set until about $t_f = 1.2$. Past this point double precision is not adequate for this problem. We have also controlled this system with the same closed loop Lyapunov exponents, but the principal directions for the largest and smallest exponents were switched. Results were quite similar to the unswitched case.

We have also controlled the Duffing problem in the chaotic region

$$\dot{x} = v,$$
$$\dot{v} = x - x^3 + 0.4\cos(t) - 0.25v + u, \qquad (32)$$

with initial conditions on the attractor of $x(t_0) = -1.241\ 861\ 655\ 22$ and $v(t_0) = 0.500\ 205\ 770\ 569$. The control was placed on the velocity state, assuming that, as a mechanical system only the velocity state could be directly influenced by an external force. Figure 2 shows the open and closed loop exponents, where closed loop exponents were chosen to be $\lambda_{i,C} = -0.1$ and -0.7. The control method was successful until about $t_f \approx 25$. As with the Lorenz problem, this is about the time that the diagonal elements of the $\mathcal{W}$ matrix span more than the 13 significant figures permitted in double precision arithmetic.

Returning to the Lorenz system, the initial conditions $x = -12.315\ 124\ 003\ 712$, $y = 0$, and $z = 51.695\ 028\ 436\ 029$ are periodic with period $\tau = 1.031\ 419\ 488\ 278$ time units. This periodic orbit, shown in Fig. 3, samples both "wings" of the "butterfly," and has open loop Poincaré exponents of $\lambda_i = 1.4625 + 0i$, $0 + 0i$, and $-22.4624 + 0i$. An attempt to move only the unstable root to a value of minus one led to an interesting insight. Expression (28) can be rewritten as

$$S(t) = \Phi_\gamma(t)\Phi_{x,C}^{-1}(t). \qquad (33)$$

Both matrices $\Phi_\gamma(t)$ and $\Phi_{x,C}(t)$ are well behaved, but the latter matrix has a dynamical mode which is decreasing in amplitude exponentially, like $\exp(-22.4t)$. This means that $S(t)$ increases from its reasonable initial conditions at a maximal rate of $\exp(+22.4t)$, and the gain

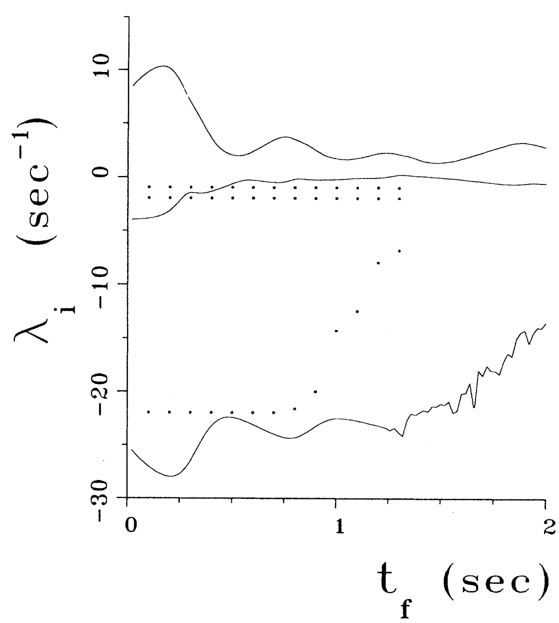

FIG. 1. Open loop (solid lines) and closed loop (dot) Lyapunov exponents for the Lorenz system, with no change in principal directions, as a function of final time.

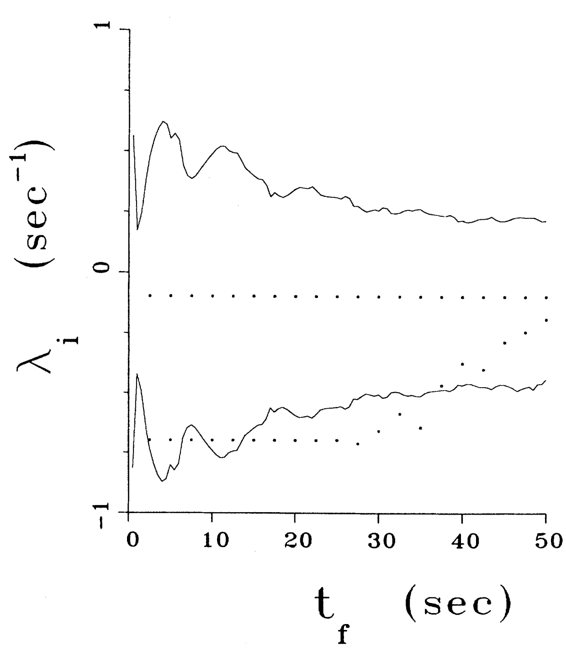

FIG. 2. Open loop (solid) and closed loop (dots) Lypaumov exponents for the Duffing oscillator as a function of final time.

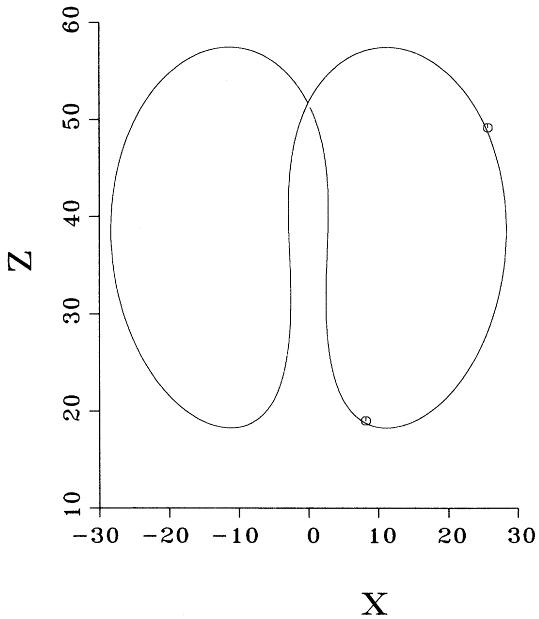

FIG. 3. An unstable periodic orbit imbedded in the Lorenz attractor seen in its x, z projection. Symbols locate the optimal positions for control.

matrix peaks *very* sharply near t_f. It was therefore necessary to change the most negative Poincaré exponent as well, and new values of $\lambda_i = -1+0i$, $-2+0i$, and $-3+0i$ were chosen.

Figure 4 shows the three elements of the gain matrix $G(t)$ over one period. (The gain matrix has units which depend on which particular element is referred to, so no units are shown in the figure.) It is obvious in one sense that $\mathbf{u}^T C \mathbf{u}$ has been minimized by reducing the control usage to sharp spikes. (In fact, the gain spikes at $t \approx 0.4$ have been truncated to show detail in the rest of the function. They really extend to $G_{ij} \approx 10^5$.) The location of these moments of intense control are shown in Fig. 3 as symbols on the periodic orbit. The fact that the gain functions are well approximated by δ functions indicates that the optimal control is almost an OGY control. However, this method locates the optimal positions for the required surfaces of section, as well as allowing complete specification of the stability properties. These same two positions on the periodic orbit also appear when the closed loop exponents are set to other values.

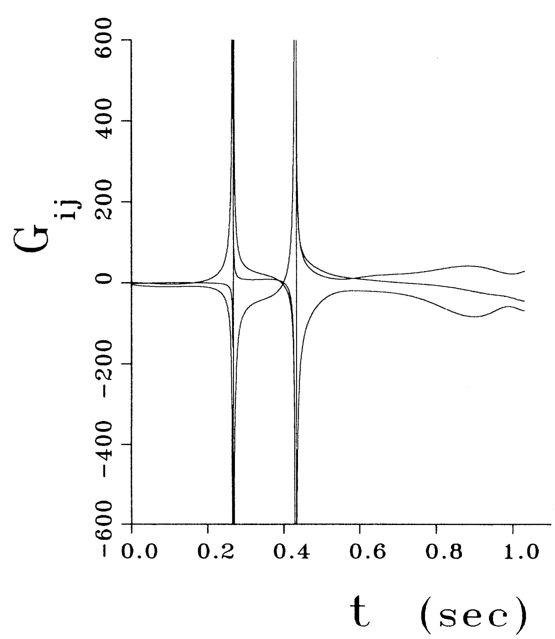

FIG. 4. Gain matrix elements $G_{ij}(t)$ for a controlled periodic orbit in the Lorenz system.

Finally, to consider a larger order system, the restricted problem of three bodies from orbital mechanics was chosen. This represents the motion of a virtually massless particle in the gravitational field of two massive bodies executing circular motion about their common center of mass. The massive objects (the primaries) have masses $1-\mu$ and μ in the usual dimensionless coordinates [9]. Without control, this conservative system has Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + p_x y - p_y x - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{34}$$

where

$$r_1^2 = (x-\mu)^2 + y^2 + z^2, \quad r_2^2 = (x+1-\mu)^2 + y^2 + z^2. \tag{35}$$

The mass parameter was chosen to be $\mu = 1/3$, and initial conditions $x = 0.035\,899\,801\,22$, $p_y = -1.397\,699\,585\,89$ with all other initial states zero, are periodic with a period of $\tau = 3.445\,161\,232\,351$. Since the momenta are the inertial velocity components, allowing the spacecraft to accelerate in any direction corresponds to placing control terms on all three $\dot{p}_i$ equations of motion. Partitioning the state vector as $\mathbf{X}^T = (x,\ y,\ z,\ p_x,\ p_y,\ p_z)$ and choosing the B, C matrices as

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{36}$$

implements this control, and specifies that the average squared spacecraft acceleration due to control over one orbit is to be minimized. Open loop Poincaré exponents are $0 \pm 0.7753i$, $0 \pm 6641i$, and $0 \pm 0i$, so this orbit is linearly stable, or at least as stable as a Hamiltonian system can be.

Control was used to set the closed loop Poincaré exponents to $-0.2\pm0.7753i$, $-0.3\pm0.6641i$, and $-0.5\pm0i$. The eigenvector matrix $E(t_0)$ was not altered. (Of course, care must be exercised in choosing new exponents that the resulting closed loop matrix $\Phi_{x,C}$ will be real.) The eighteen elements of the gain matrix $G(t)$ are shown as functions of time in Fig. 5. No large spikes are apparent, although the gain matrix $G(t)$ is clearly not periodic. (No

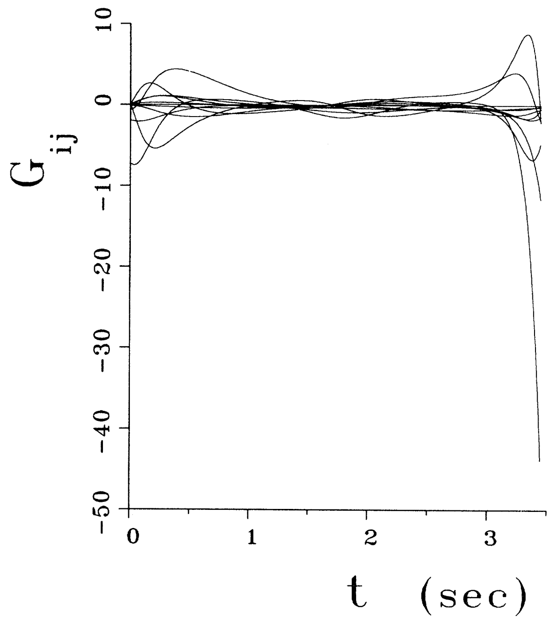

FIG. 5. Gain matrix elements $G_{ij}(t)$ for a controlled periodic orbit in the restricted problem of three bodies.

units are shown for the G_ij components, since the gain matrix is a mixed-unit quantity.)

V. DISCUSSION AND CONCLUSIONS

In this paper we have reported a method to *a priori* dictate all of the closed loop Lyapunov or Poincaré exponents and all of the principal dynamical directions of a dynamical system, while minimizing a positive definite function of the control. The development leads to a linear boundary value problem, which can be solved in one iteration. It has been successfully applied to several example problems, and both stability exponents and principal directions can be dictated at will.

The limitation to a finite time interval is an unavoidable consequence of the chaotic nature of the underlying dynamics. Since there are instabilities, Φ_x will grow unbounded, and the system cannot be reliably predicted for an unbounded time period. However, with the freedom to set both closed loop stability exponents *and* principal dynamical directions, control over one time interval t_1, t_2 can be spliced into control over the next time interval t_2, t_3. The method can thus be extended to an arbitrarily long interval of time. Admittedly, this will only be feasible in problems where the required computation can be completed faster than real time. One example of such a problem is the multiple encounter mission of the Galileo spacecraft at Jupiter, where individual orbits will take months of real time. An exception is the case of periodic orbits, where the gain matrix can be precomputed and used over and over again for each period. Only the speed at which the system can retrieve the gain matrix then limits the applicability of the method.

Also, several extensions to this method are also easily done. In the case of the thruster controlled spacecraft, it seems more natural to the author to minimize the total change in spacecraft velocity due to control, and thus minimize the fuel usage. But this implies a cost function

$$\mathcal{J} = \frac{1}{2}\int_{t_0}^{t_f} \left\{\mathbf{u}^T C \mathbf{u}\right\}^{1/2} dt, \tag{37}$$

with the same B and C matrices used earlier. This leads to a nonlinear boundary value problem, which would be solved by an iterative technique.

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